Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

Lecture 12A. 3D shell finite element

05.2025

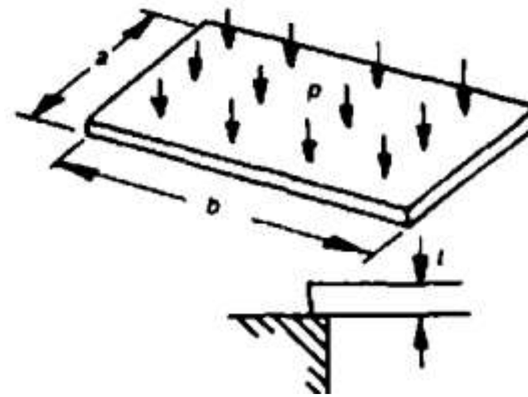
Shells and plates

Thin-shells and plates models can be applied to analyze the following constructions:

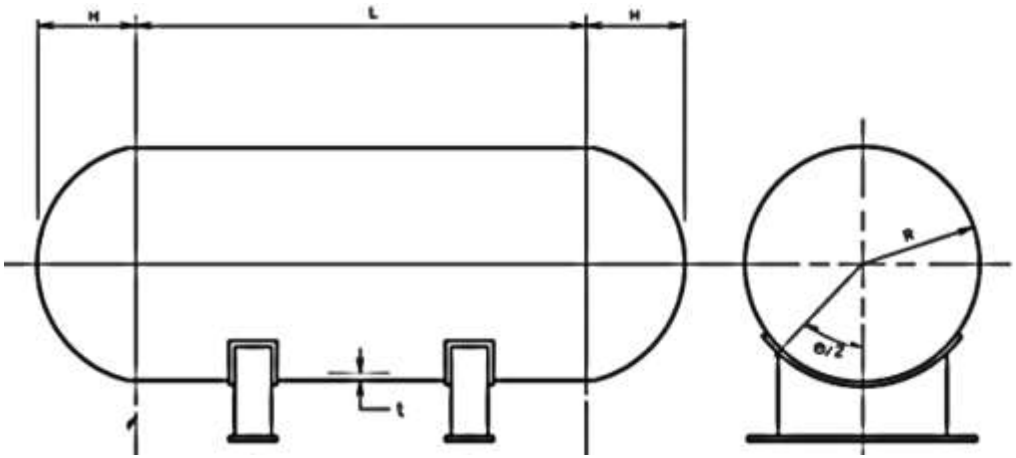
aircraft fusalage, the wing cover,

boat hull,

roof (floor) of building.



Rectangular plate



Thin shell of revolution

Examples of plates and shells

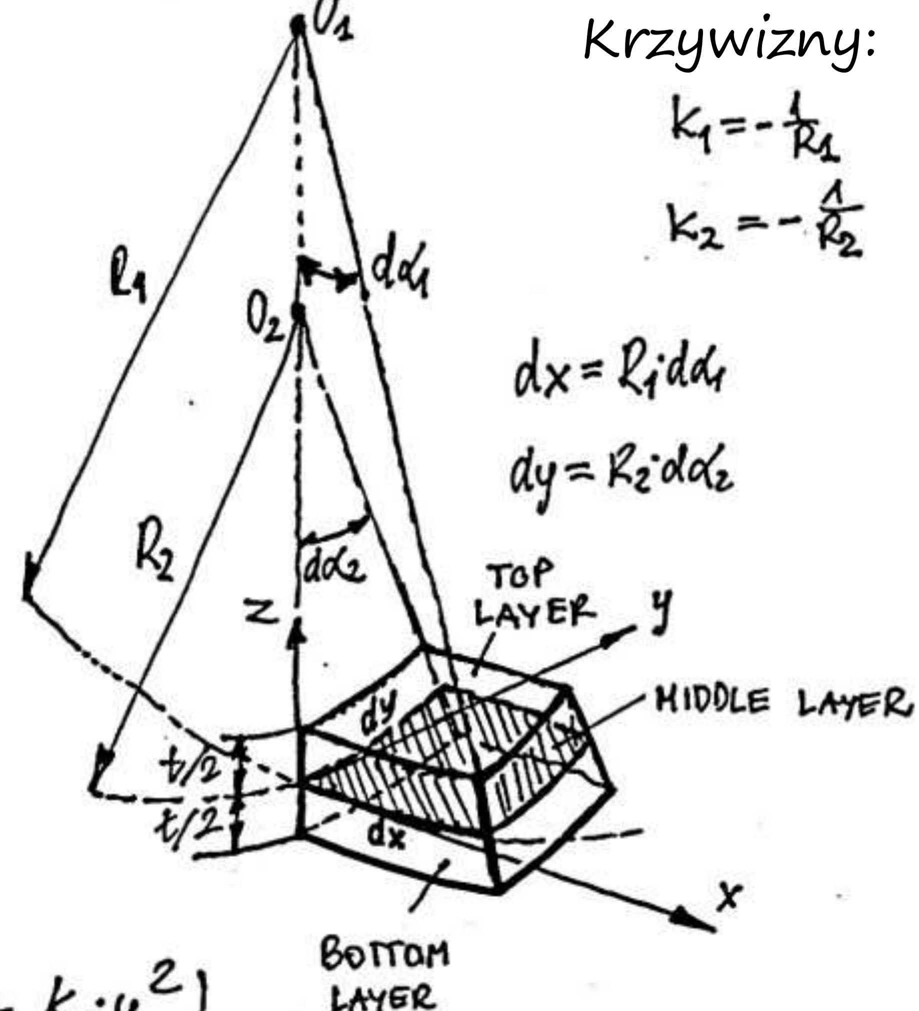
aircraft skins (shell model)

construction of a building (plate model)

a motor yacht:
the hull (shell), the deck (plate)

Linear theory of thin shells

Types of shells:
- eliptical,
- cylindrical,
- spherical,
- toroidal,
- hyperbolic.



Krzywizny:

$$k_1 = -\frac{1}{R_1}$$

$$k_2 = -\frac{1}{R_2}$$

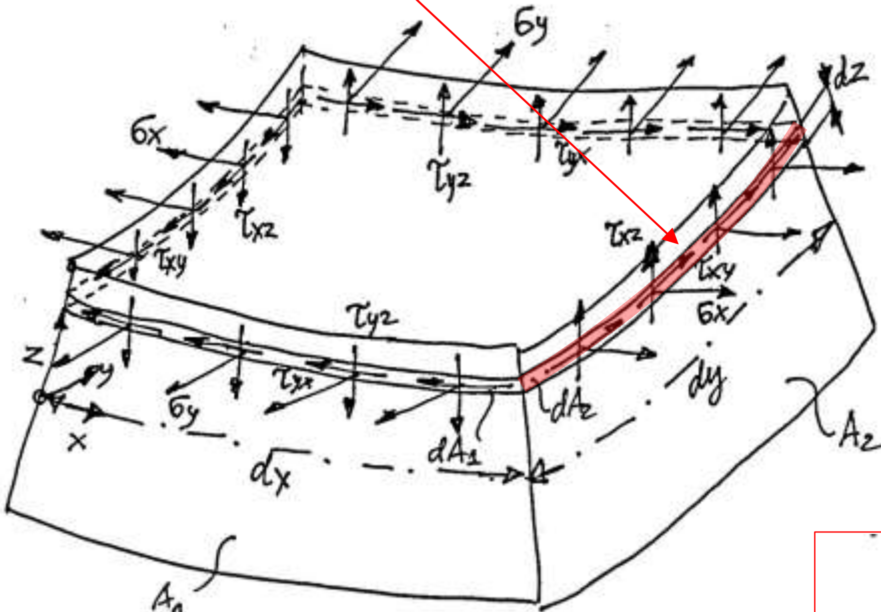
$$dx = R_1 \cdot d\alpha_1$$

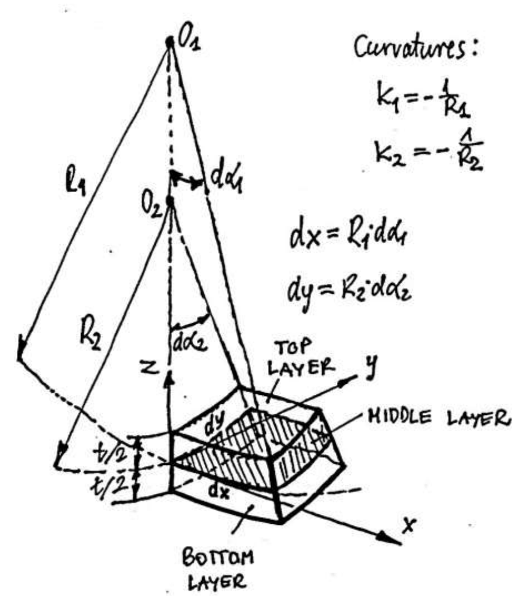
$$dy = R_2 \cdot d\alpha_2$$

$$z = 0.5\left(k_1 \cdot x^2 + 2k_{12} \cdot xy + k_2 \cdot y^2\right)$$

Internal force at level z in a small area dA_z

$$\sigma_x \cdot \boxed{dz \cdot (R_2 - z)\, d\alpha_2} = \sigma_x \cdot dz \left(1 - \frac{z}{R_2}\right) R_2\, d\alpha_2 =$$
$$= \sigma_x \left(1 - \frac{z}{R_2}\right) dz\, dy$$





Internal force at level z in a small area dA_z per unit length

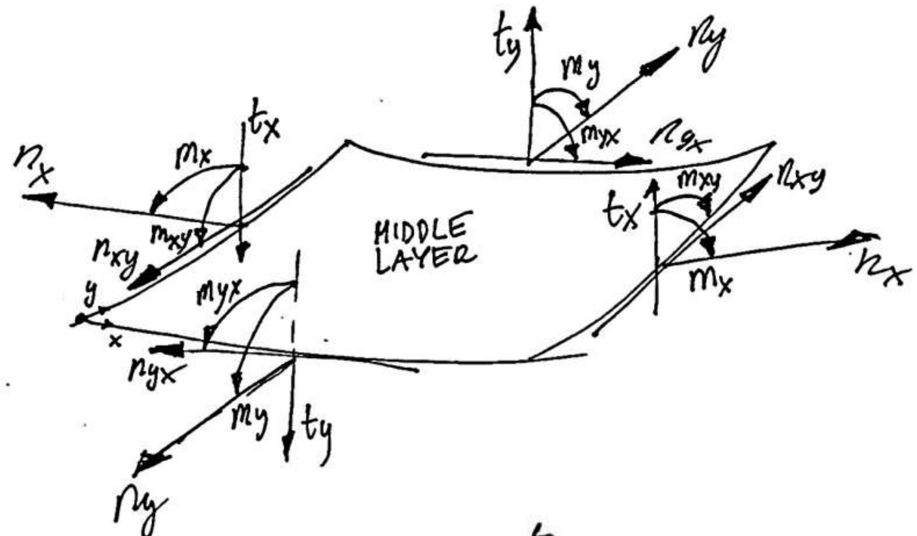
$$dn_x = \frac{\sigma_x \left(1 - \frac{z}{R_2}\right) dz\, dy}{dy} = \sigma_x \left(1 - \frac{z}{R_2}\right) dz$$

Internal force on a crosssectional area Az per unit length:

$$\boxed{n_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x \left(1 - \frac{z}{R_2}\right) dz} \qquad \left(\frac{N}{m}\right)$$

Assuming: $\frac{z}{R_1} \approx 0$, $\frac{z}{R_2} = 0$ $\Rightarrow$ $n_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x \, dz$

Internal forces:



n- normal force per unit length

m- bending moment per unit length

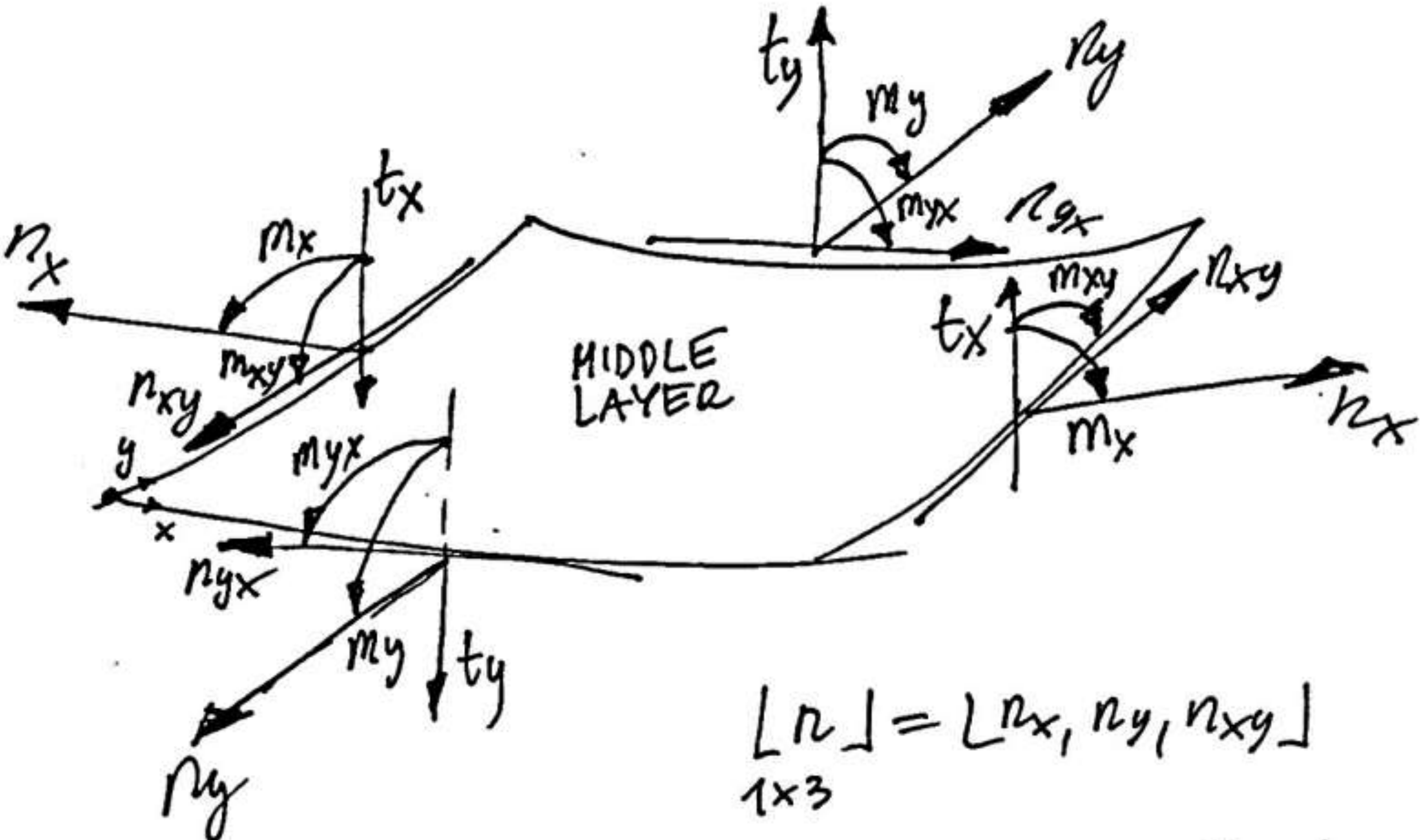
t- shear force per unit length

$$n_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x \, dz \quad , \quad n_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_y \, dz \quad , \quad n_{xy} = n_{yx} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xy} \, dz$$

$$m_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x \cdot z \, dz \quad , \quad m_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_y \cdot z \, dz \quad , \quad m_{xy} = m_{yx} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xy} \, z \, dz \quad \left(\frac{Nm}{m}\right)$$

$$t_x = \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y} \quad , \quad t_y = \frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x}$$

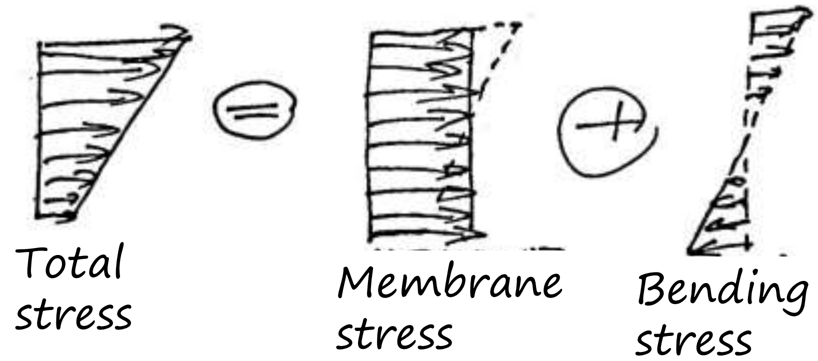
Internal forces:

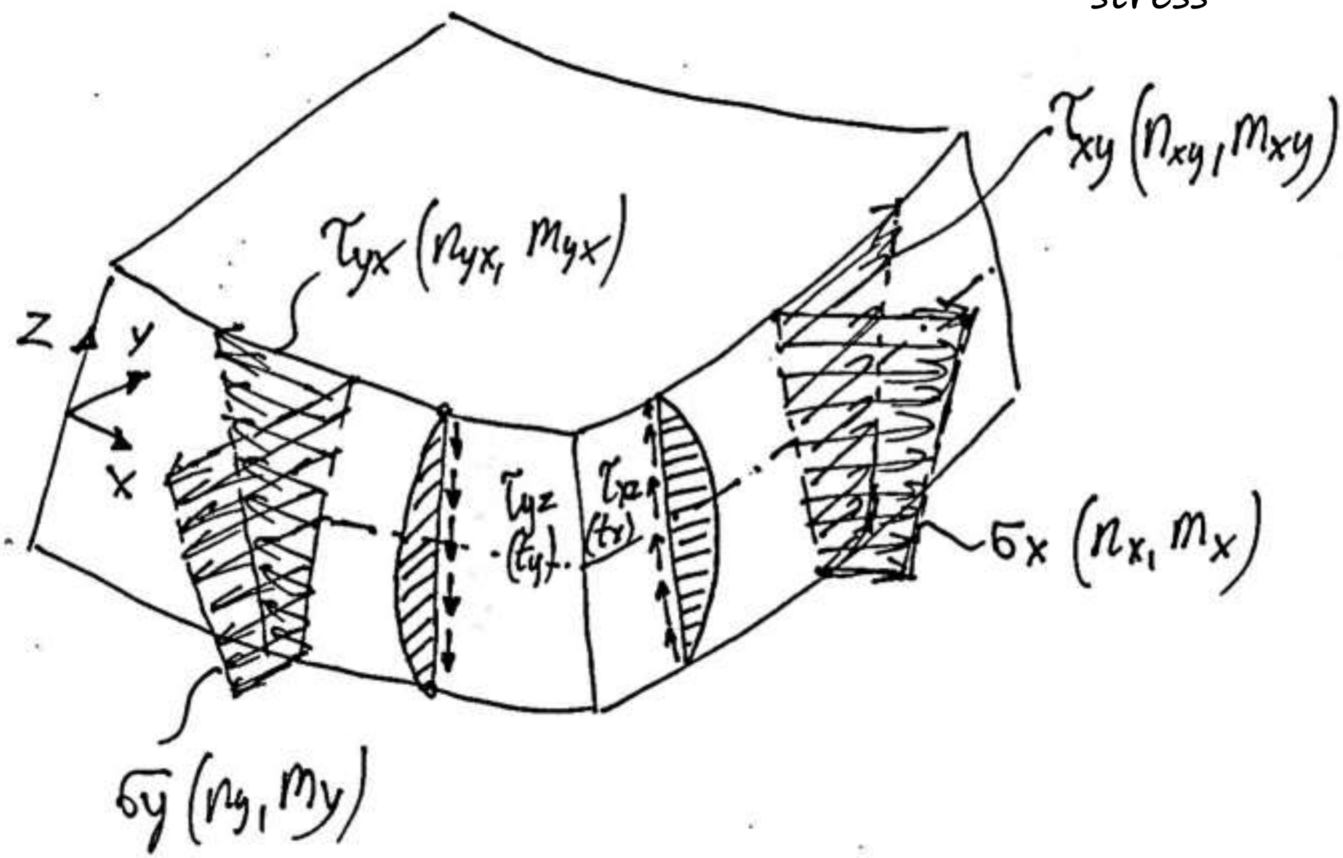


$$\lfloor n \rfloor_{1\times 3} = \lfloor n_x, n_y, n_{xy} \rfloor$$

$$\lfloor m \rfloor_{1\times 3} = \lfloor m_x, m_y, m_{xy} \rfloor$$

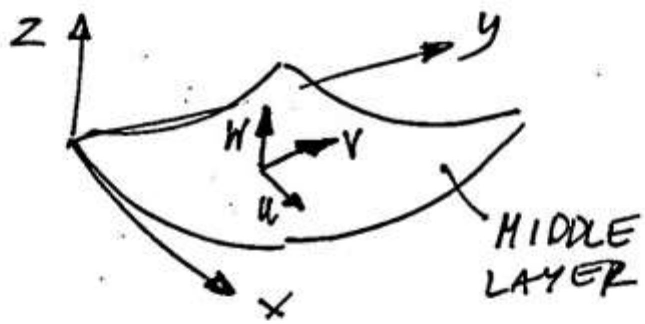
Stress components





Membrane strain:

1) Deformation of a middle layer in xy plane



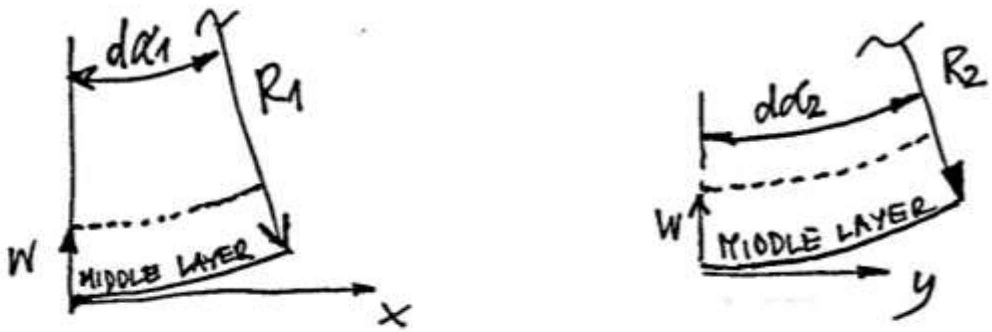
1°)

$$\varepsilon_y^{1^\circ} = \frac{\partial v}{\partial y}$$

$$\varepsilon_x^{1^\circ} = \frac{\partial u}{\partial x}$$

Membrane strain :

2) Deformation of a middle layer along z axis



$$\varepsilon_x^{2°} = \frac{(R_1 - w)\,d\alpha_1 - R_1\,d\alpha_1}{R_1\,d\alpha_1} = -\frac{W}{R_1} = k_1 \cdot W$$

$$\varepsilon_y^{2°} = \frac{(R_2 - w)\,d\alpha_2 - R_2\,d\alpha_2}{R_2\,d\alpha_2} = -\frac{W}{R_2} = k_2 \cdot W$$

Curvatures due to geometry

MEMBRANE STRAIN:

$$1^\circ) + 2^\circ)$$

$$\varepsilon_x^{MID} = \frac{\partial u}{\partial x} + k_1 \cdot w$$

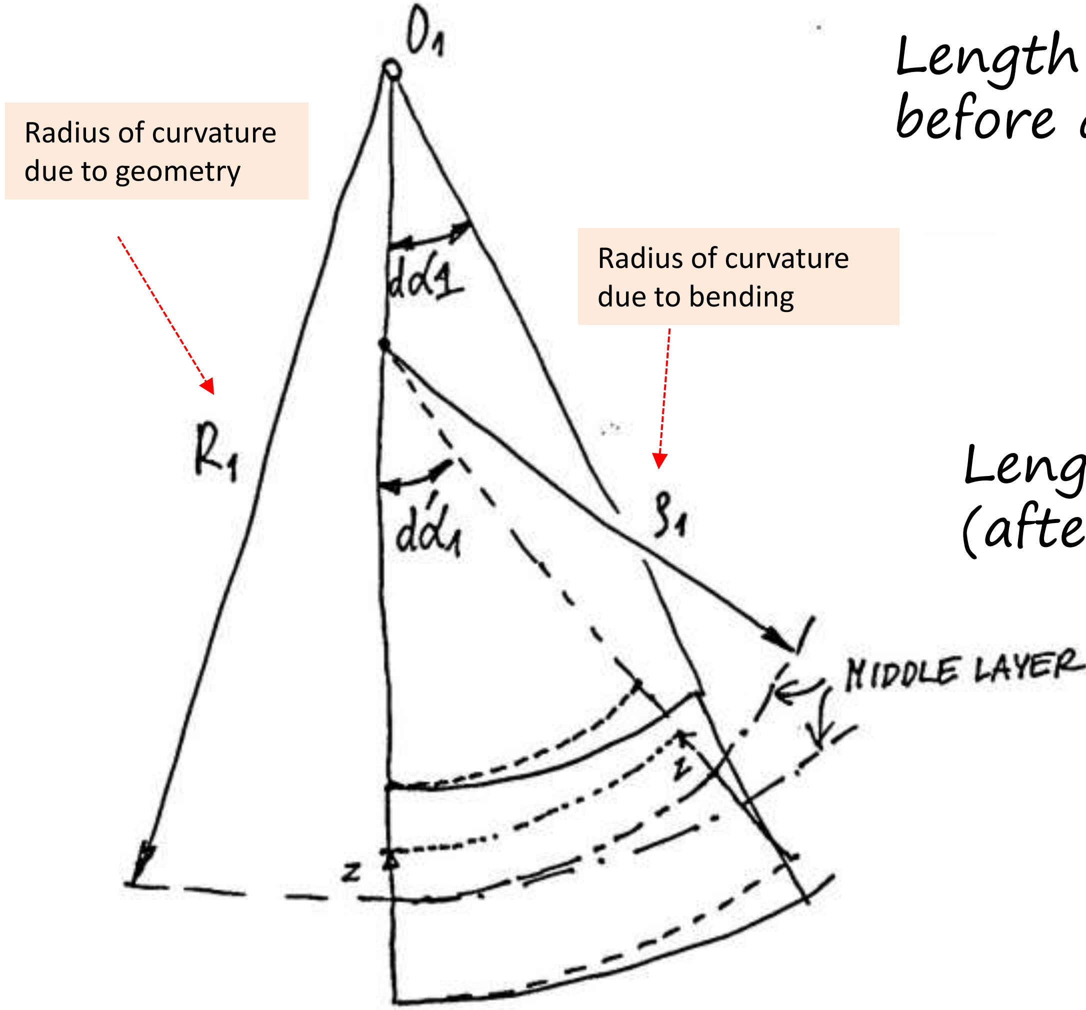
$$\varepsilon_y^{MID} = \frac{\partial v}{\partial y} + k_2 \cdot w$$

$$\gamma_{xy}^{MID} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + 2k_{12} \cdot w$$

Curvatures due to geometry

Bending strain:

3) Deformation of the layer at level z



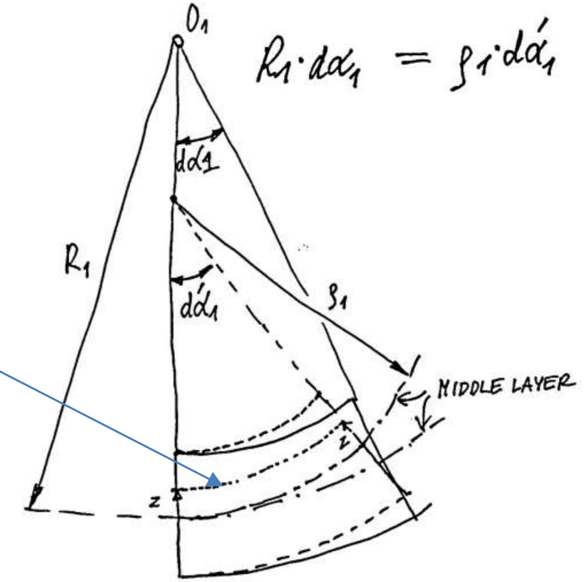
Length of the layer at level z before deformation:

$$R_1\left(1-\frac{z}{R_1}\right)\cdot d\alpha_1$$

Length of the middle layer (after and before deformation)

$$R_1\cdot d\alpha_1 = \rho_1\cdot d\alpha_1'$$

Bending strain
(deformation of the layer at level z)



$$\varepsilon_x(z) = \frac{\rho_1\left(1-\frac{z}{\rho_1}\right)d\alpha_1' - R_1\left(1-\frac{z}{R_1}\right)d\alpha_1}{R_1\left(1-\frac{z}{R_1}\right)d\alpha_1} =$$

$$= \frac{\rho_1\left(1-\frac{z}{\rho_1}\right)\frac{R_1}{\rho_1}d\alpha_1 - R_1\left(1-\frac{z}{R_1}\right)d\alpha_1}{R_1\left(1-\frac{z}{R_1}\right)d\alpha_1} = \frac{\left(1-\frac{z}{\rho_1}\right)-\left(1-\frac{z}{R_1}\right)}{\left(1-\frac{z}{R_1}\right)} =$$

$$= \underbrace{\frac{1-\frac{z}{\rho_1}}{1-\frac{z}{R_1}}}_{\approx 1} - 1 = -\frac{z}{\rho_1} = -\frac{\partial^2 w}{\partial x^2}\cdot z = \varkappa_x \cdot z$$

Curvature due to bending

$$\varepsilon_y(z) = -\frac{\partial^2 w}{\partial y^2}\cdot z = \varkappa_y \cdot z \quad ; \quad \gamma_{xy}(z) = -2\frac{\partial^2 w}{\partial x \partial y}\cdot z = \varkappa_{xy}\cdot z$$

Total strain component vector (Membrane + bending)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^{MID} \\ \varepsilon_y^{MID} \\ \gamma_{xy}^{MID} \end{Bmatrix} + \begin{Bmatrix} \varepsilon_x(z) \\ \varepsilon_y(z) \\ \gamma_{xy}(z) \end{Bmatrix} = \underset{3\times 1}{\{\varepsilon^{MID}\}} + \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \cdot z \Rightarrow \underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 1}{\{\varepsilon^{MID}\}} + \underset{3\times 1}{\{\kappa\}} \cdot z$$

Curvature due to bending

stress component vector

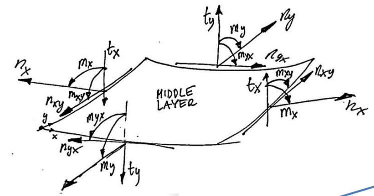
Assuming plain stress:

$$\underset{3\times 3}{[D]} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & & \frac{1}{2}(1-\nu) \end{bmatrix}$$

We have:

$$\underset{3\times 1}{\{\sigma\}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \boxed{\underset{3\times 3}{[D]} \cdot \underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 3}{[D]} \cdot \underset{3\times 1}{\{\varepsilon^{MID}\}} + \underset{3\times 3}{[D]} \underset{3\times 1}{\{\kappa\}} \cdot z}$$

Internal forces:



$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \underset{3\times3}{[D]}\cdot\underset{3\times1}{\{\varepsilon^{MID}\}} + \underset{3\times3}{[D]}\underset{3\times1}{\{\kappa\}}\cdot z$$

$$\underset{3\times1}{\{n\}} = \begin{Bmatrix}n_x\\ n_y\\ n_{xy}\end{Bmatrix} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} dz = \boxed{t\underset{3\times3}{[D]}\cdot\underset{3\times1}{\{\varepsilon^{MID}\}}} + \begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix} = \underset{3\times3}{[D_n]}\cdot\underset{3\times1}{\{\varepsilon^{MID}\}}$$

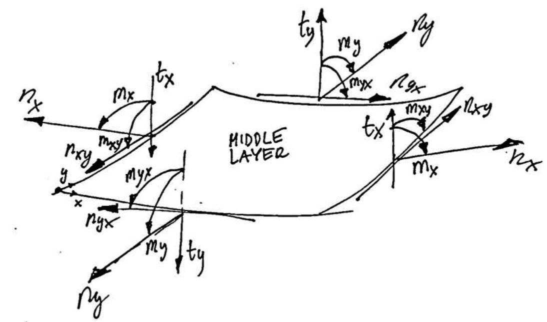
$$\underset{3\times1}{\{m\}} = \begin{Bmatrix}m_x\\ m_y\\ m_{xy}\end{Bmatrix} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}\cdot z\, dz = \begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix} + \boxed{\frac{t^3}{12}\underset{3\times3}{[D]}\cdot\{\kappa\}} =$$

$$= \frac{Et^3}{12(1-\nu^2)}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\cdot\begin{Bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{Bmatrix} = \underset{3\times3}{[D_m]}\cdot\underset{3\times1}{\{\kappa\}}$$

STRESS COMPONENTS AS FUNCTIONS OF INTERNAL FORCES

$$\boxed{\underset{3\times1}{\{\sigma\}} = \underset{3\times3}{[D]}\,\underset{3\times1}{\{\varepsilon^{MID}\}} + \underset{3\times3}{[D]}\cdot\{\kappa\}\cdot z} = \frac{1}{t}\cdot\underset{3\times1}{\{n\}} + \frac{12}{t^3}\{m\}\cdot z$$

$$\underset{3\times3}{[D]}\,\underset{3\times1}{\{\varepsilon^{MID}\}} = \boxed{\frac{1}{t}\cdot\underset{3\times3}{[D]}^{-1}\cdot\underset{3\times1}{\{n\}}} \qquad \underset{3\times3}{[D]}\cdot\{\kappa\} = \boxed{\frac{12}{t^3}\underset{3\times3}{[D]}^{-1}\cdot\underset{3\times1}{\{m\}}}$$



normal stresses :

$$\sigma_x = \frac{n_x}{t} + \frac{12\,m_x}{t^3}\cdot z$$

$$\sigma_y = \frac{n_y}{t} + \frac{12\,m_y}{t^3}\cdot z$$

shear stresses:

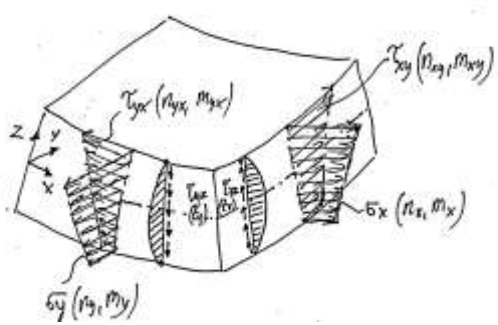
$$\tau_{xy} = \tau_{yx} = \frac{n_{xy}}{t} + \frac{12\,m_{xy}}{t^3}\cdot z$$

$$\tau_{xz} = \frac{3\,t_x}{2t}\left(1 - \frac{4z^2}{t^2}\right)$$

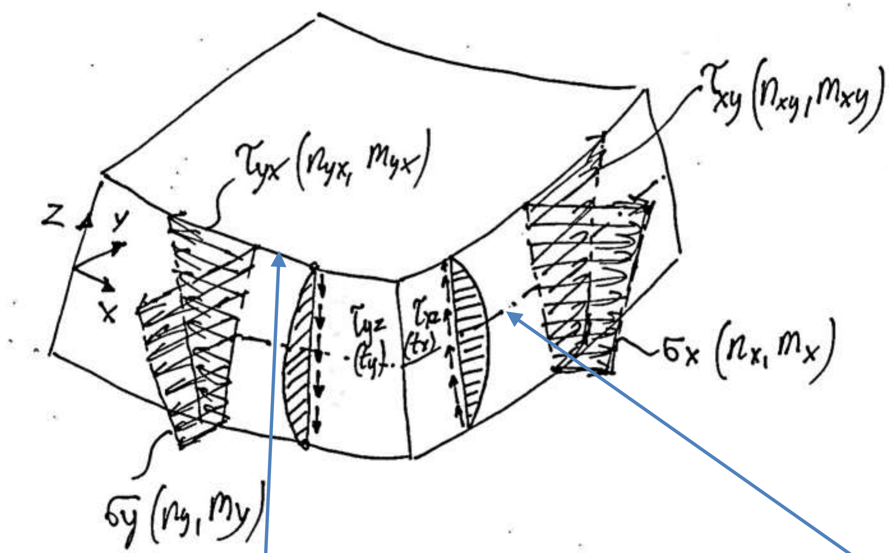
$$\tau_{yz} = \frac{3\,t_y}{2t}\left(1 - \frac{4z^2}{t^2}\right)$$

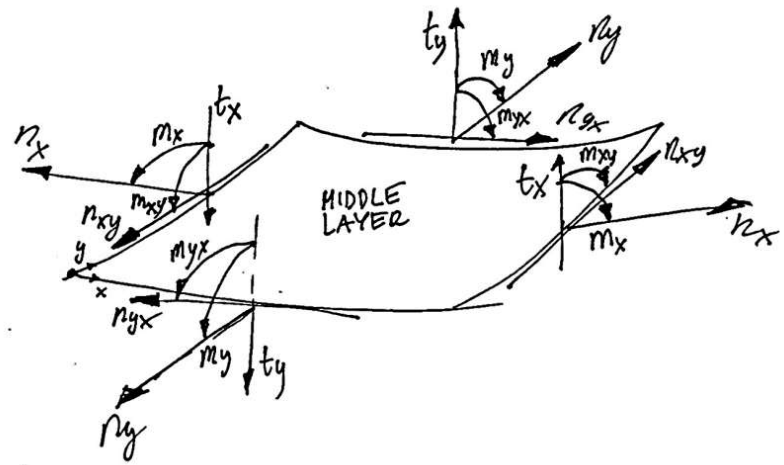
$$t_x = \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y}$$

$$t_y = \frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x}$$



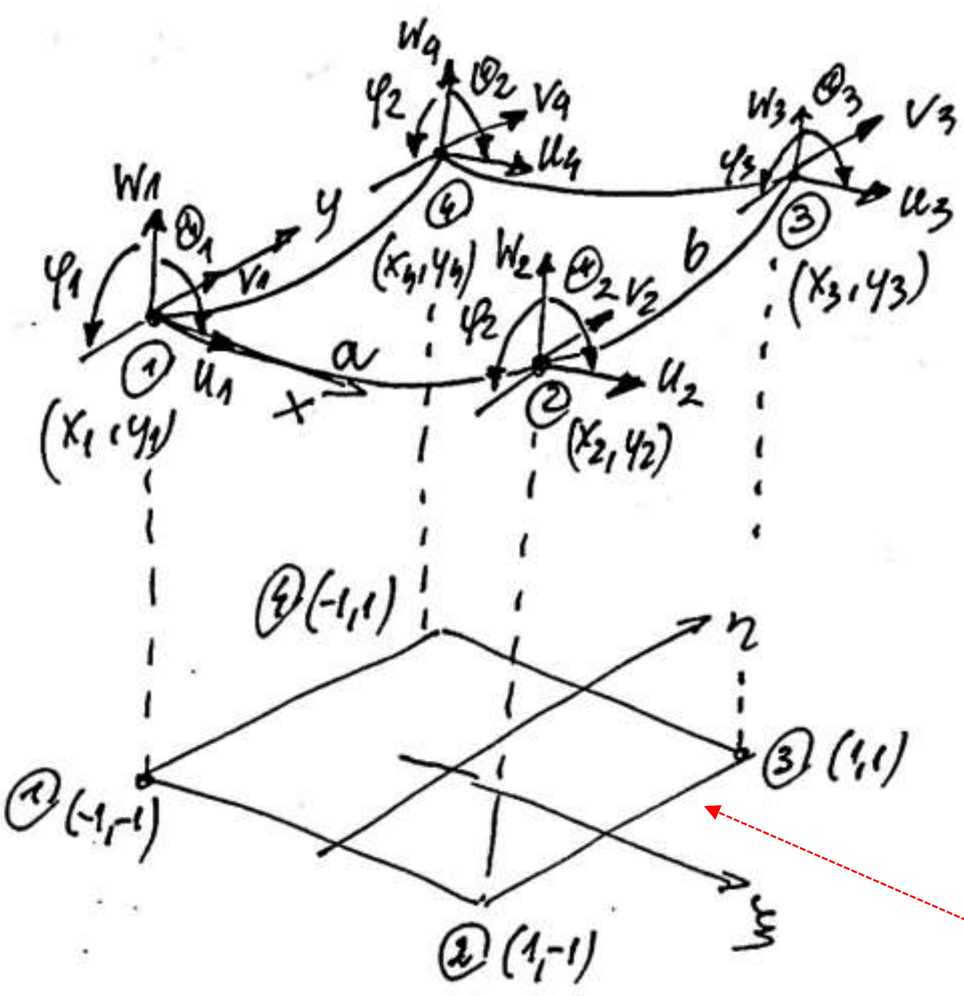
MAXIMUM VALUES OF STRESS COMPONENTS





TOP LAYER	MIDDLE LAYER
$\sigma_x^{TOP} = \frac{n_x}{t} + \frac{6m_x}{t^2}$	$\sigma_x^{MID} = \frac{n_x}{t}$
$\sigma_y^{TOP} = \frac{n_y}{t} + \frac{6m_y}{t^2}$	$\sigma_y^{MID} = \frac{n_y}{t}$
$\tau_{xy}^{TOP} = \frac{n_{xy}}{t} + \frac{6m_{xy}}{t^2}$	$\tau_{xy}^{MID} = \frac{n_{xy}}{t}$
$\tau_{xz}^{TOP} = 0$	$\tau_{xz}^{MID} = \frac{3}{2} \cdot \frac{t_x}{t}$
$\tau_{yz}^{TOP} = 0$	$\tau_{yz}^{MID} = \frac{3}{2} \cdot \frac{t_y}{t}$

An isoparametric shell finite element



Parent element

$$n = 4$$

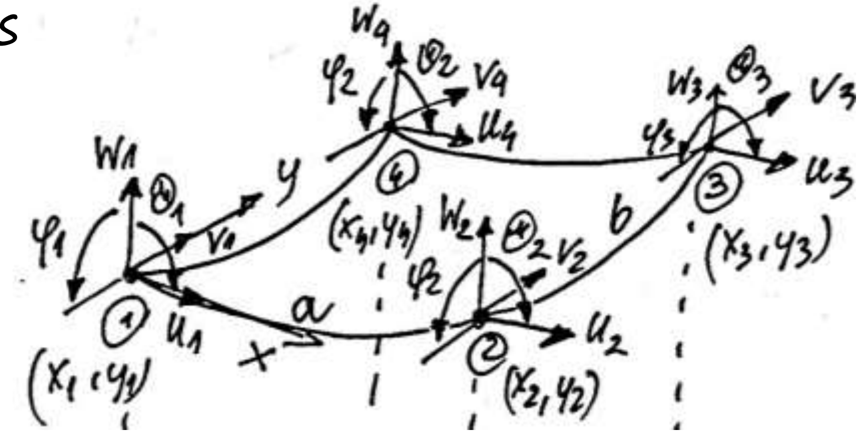
$$\boxed{n_p = 5} \Rightarrow$$

$$n_e = 4 \cdot 5 = 20$$

$$\varphi_i = \left.\frac{dw}{\partial y}\right|_i$$

$$\theta_i = -\left.\frac{\partial w}{\partial x}\right|_i$$

Local vector of nodal parameters (three parts)



$$\underset{1\times 4}{\lfloor q_u \rfloor_e} = \lfloor u_1, u_2, u_3, u_4 \rfloor_e$$

$$\underset{1\times 4}{\lfloor q_v \rfloor_e} = \lfloor v_1, v_2, v_3, v_4 \rfloor_e$$

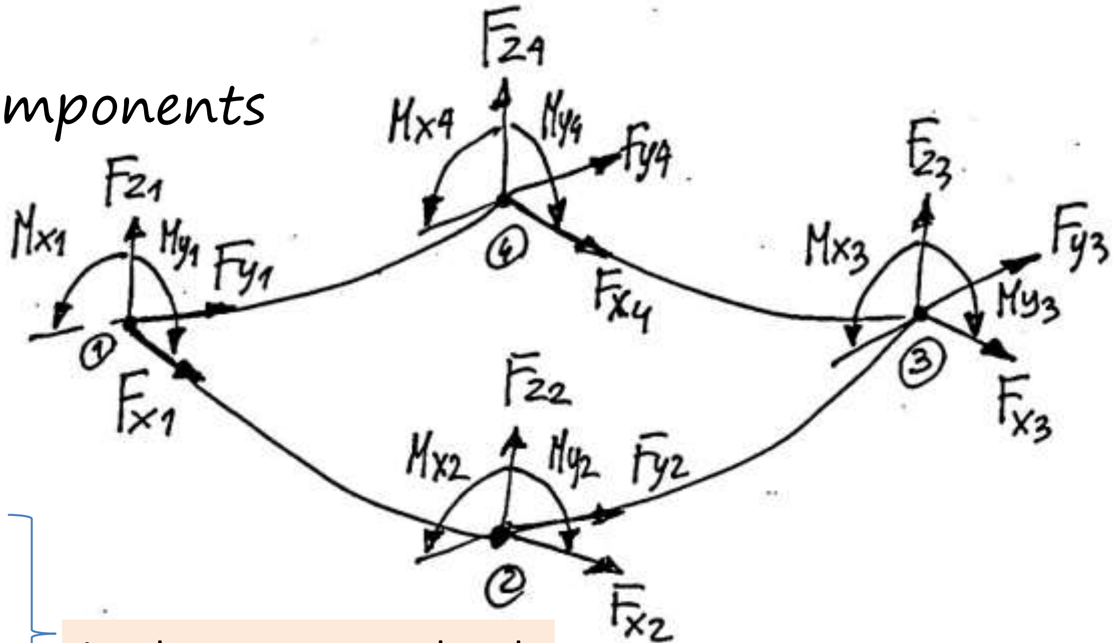
degrees of freedom related to deformations in the element plane

$$\underset{1\times 12}{\lfloor q_w \rfloor_e} = \lfloor w_1, \varphi_1, \Theta_1, w_2, \varphi_2, \Theta_2, w_3, \varphi_3, \Theta_3, w_4, \varphi_4, \Theta_4 \rfloor_e$$

degrees of freedom related to out-of-plane deformations of the element

$$\boxed{\underset{1\times 20}{\lfloor q \rfloor_e} = \lfloor \lfloor q_u \rfloor_e, \lfloor q_v \rfloor_e, \lfloor q_w \rfloor_e \rfloor_e}$$

Local vector of load components (three parts)



$$\underset{1\times 4}{\lfloor F_x \rfloor_e} = \lfloor F_{x1}, F_{x2}, F_{x3}, F_{x4} \rfloor$$

$$\underset{1\times 4}{\lfloor F_y \rfloor_e} = \lfloor F_{y1}, F_{y2}, F_{y3}, F_{y4} \rfloor$$

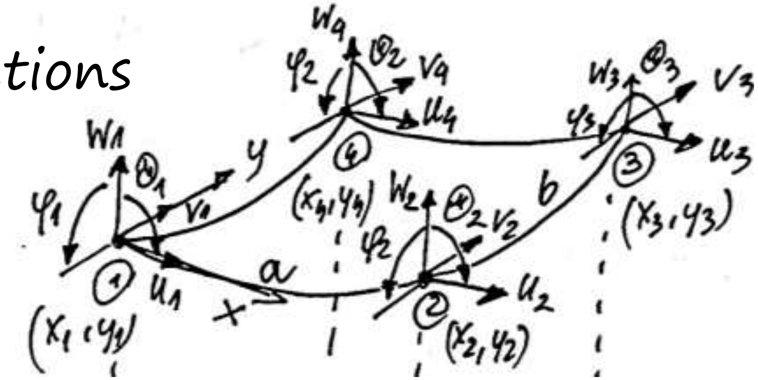
Load components related to deformations in the element plane

$$\underset{1\times 12}{\lfloor F_z \rfloor_e} = \lfloor F_{z1}, M_{x1}, M_{y1}, F_{z2}, M_{x2}, M_{y2}, F_{z3}, M_{x3}, M_{y3}, F_{z4}, M_{x4}, M_{y4} \rfloor$$

Load components related to out-of-plane deformations of the element

$$\boxed{\underset{1\times 20}{\lfloor F \rfloor_e} = \lfloor \lfloor F_x \rfloor_e, \lfloor F_y \rfloor_e, \lfloor F_z \rfloor_e \rfloor_e}$$

Nodal approximation and shape functions



$$u = N_1 \cdot u_1 + N_2 \cdot u_2 + N_3 \cdot u_3 + N_4 \cdot u_4$$

$$v = N_1 \cdot v_1 + N_2 \cdot v_2 + N_3 \cdot v_3 + N_4 \cdot v_4$$

Displacements in the element plane

$$w = N_{11} \cdot w_1 + N_{12} \cdot \varphi_1 + N_{13} \cdot \Theta_1 + N_{21} w_2 + N_{22} \cdot \varphi_2 + N_{23} \Theta_2 + $$
$$+ N_{31} w_3 + N_{32} \varphi_3 + N_{33} \Theta_3 + N_{41} w_4 + N_{42} \varphi_4 + N_{43} \Theta_4$$

Displacements out of the element plane

$$\underset{1\times 4}{\lfloor N \rfloor} = \lfloor N_1, N_2, N_3, N_4 \rfloor$$

(polynomials of ξ *and* η)

$$\underset{1\times 12}{\lfloor N_w \rfloor} = \lfloor N_{11}, N_{12}, N_{13}, N_{21}, N_{22}, N_{23}, N_{31}, N_{32}, N_{33}, N_{41}, N_{42}, N_{43} \rfloor$$

(Hermite polynomials)

Nodal approximation and shape functions

$$u = \underset{1\times4}{\lfloor N \rfloor} \cdot \{q_u\}_e$$

$$v = \underset{1\times4}{\lfloor N \rfloor} \{q_v\}_e$$

$$w = \underset{1\times12}{\lfloor N_w \rfloor} \{q_w\}_e$$

Displacements in the element plane

Displacements out of the element plane

$$\underset{3\times1}{\{u\}} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \underset{3\times20}{\begin{bmatrix} \underset{1\times4}{\lfloor N \rfloor} & \underset{1\times4}{\lfloor 0 \rfloor} & \underset{1\times12}{\lfloor 0 \rfloor} \\ \underset{1\times4}{\lfloor 0 \rfloor} & \underset{1\times4}{\lfloor N \rfloor} & \underset{1\times12}{\lfloor 0 \rfloor} \\ \underset{1\times4}{\lfloor 0 \rfloor} & \underset{1\times4}{\lfloor 0 \rfloor} & \underset{1\times12}{\lfloor N_w \rfloor} \end{bmatrix}} \cdot \underset{20\times1}{\{q\}_e} = \underset{3\times20}{[N]} \cdot \underset{20\times1}{\{q\}_e}$$

Membrane strain

Vector of degrees of freedom related to deformations in the element plane

$$\varepsilon_x^{MID} = \frac{\partial u}{\partial x} + k_1 \cdot W = \frac{\partial \lfloor N \rfloor_{1\times 4}}{\partial x} \cdot \{q_u\}_e + k_1 \cdot \lfloor N_W \rfloor_{1\times 12} \cdot \{q_W\}_{e\,12\times 1}$$

$$\varepsilon_y^{MID} = \frac{\partial v}{\partial y} + k_2 \cdot W = \frac{\partial \lfloor N \rfloor_{1\times 4}}{\partial y} \{q_v\}_e + k_2 \cdot \lfloor N_W \rfloor_{1\times 12} \cdot \{q_W\}_{e\,12\times 1}$$

Curves corresponding to geometry

Vector of degrees of freedom associated with out-of-plane deformations of the element

$$\gamma_{xy}^{MID} = \frac{\partial u}{\partial v} + \frac{\partial v}{\partial x} + k_{12} \cdot W$$

$$= \frac{\partial \lfloor N \rfloor_{1\times 4}}{\partial y} \cdot \{q_u\}_e + \frac{\partial \lfloor N \rfloor_{1\times 4}}{\partial x} \{q_v\}_e + k_{12} \lfloor N_W \rfloor_{1\times 12} \cdot \{q_W\}_{e\,12\times 1}$$

Bending strain (function of curvatures):

$$\varkappa_x = -\frac{\partial^2 w}{\partial x^2} = -\frac{\partial^2 \underset{1\times 12}{\lfloor N_w \rfloor}}{\partial x^2} \cdot \underset{12\times 1}{\{q_w\}_e}$$

$$\varkappa_y = -\frac{\partial^2 w}{\partial y^2} = -\frac{\partial^2 \underset{1\times 12}{\lfloor N_w \rfloor}}{\partial y^2} \cdot \{q_w\}_e$$

$$\varkappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} = -\frac{\partial^2 \underset{1\times 12}{\lfloor N_w \rfloor}}{\partial x\, \partial y} \{q_w\}_e$$

Vector of degrees of freedom associated with out-of-plane deformations of the element

STRAIN- DISPLACEMENT MATRIX

$$
\underset{6\times1}{\begin{Bmatrix} \underset{3\times1}{\{\varepsilon\}^{MID}} \\ \underset{3\times1}{\{\kappa\}} \end{Bmatrix}}
=
\underset{6\times20}{\begin{bmatrix}
\frac{\partial}{\partial x}\underset{1\times4}{\lfloor N \rfloor} & \underset{1\times4}{\lfloor 0 \rfloor} & K_1 \underset{1\times12}{\lfloor N_w \rfloor} \\
\underset{1\times4}{\lfloor 0 \rfloor} & \frac{\partial}{\partial y}\underset{1\times4}{\lfloor N \rfloor} & K_2 \underset{1\times12}{\lfloor N_w \rfloor} \\
\frac{\partial}{\partial y}\underset{1\times4}{\lfloor N \rfloor} & \frac{\partial}{\partial x}\underset{1\times4}{\lfloor N \rfloor} & K_{12} \underset{1\times12}{\lfloor N_w \rfloor} \\
\underset{3\times4}{[0]} & \underset{3\times4}{[0]} & \begin{matrix} \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\underset{1\times12}{\lfloor N_w \rfloor}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\underset{1\times12}{\lfloor N_w \rfloor}\right) \\ -2\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\underset{1\times12}{\lfloor N_w \rfloor}\right) \end{matrix}
\end{bmatrix}}
\cdot
\underset{20\times1}{\begin{Bmatrix} \underset{4\times1}{\{q_u\}} \\ \underset{4\times1}{\{q_v\}} \\ \underset{12\times1}{\{q_w\}} \end{Bmatrix}}
$$

STRAIN- DISPLACEMENT MATRIX

$$\underbrace{\{q_{uv}\}_e}_{8\times 1} = \left\{ \begin{array}{c} \underbrace{\{q_u\}_e}_{4\times 1} \\ \underbrace{\{q_v\}_e}_{4\times 1} \end{array} \right\}_e$$

The part related to membrane deformations from displacements in the element plane

Part related to membrane deformations from out-of-plane displacements of the element

$$\underbrace{\left\{ \begin{array}{c} \underbrace{\{\varepsilon\}^{MID}}_{3\times 1} \\ \underbrace{\{\mathcal{H}\}}_{3\times 1} \end{array} \right\}}_{6\times 1} = \left[\begin{array}{c:c} \underbrace{[B_M]}_{3\times 8} & \underbrace{[B_S]}_{3\times 12} \\ \hdashline \underbrace{[0]}_{3\times 8} & \underbrace{[B_B]}_{3\times 12} \end{array} \right] \cdot \left\{ \begin{array}{c} \underbrace{\{q_{uv}\}_e}_{8\times 1} \\ \underbrace{\{q_w\}_e}_{12\times 1} \end{array} \right\}_e$$

The part related to bending deformations from out-of-plane displacements of the element

STRAIN- DISPLACEMENT MATRIX

$$\underset{3\times 8}{[B_M]} = \left[\begin{array}{c|c} \frac{\partial}{\partial x}\underset{1\times 4}{\lfloor N \rfloor} & \underset{1\times 4}{\lfloor 0 \rfloor} \\ \hline \underset{1\times 4}{\lfloor 0 \rfloor} & \frac{\partial}{\partial y}\underset{1\times 4}{\lfloor N \rfloor} \\ \hline \frac{\partial}{\partial y}\underset{1\times 4}{\lfloor N \rfloor} & \frac{\partial}{\partial x}\underset{1\times 4}{\lfloor N \rfloor} \end{array}\right]$$

(middle layer)

$$\underset{3\times 12}{[B_S]} = \begin{bmatrix} k_1 \underset{1\times 12}{\lfloor N_w \rfloor} \\ k_2 \underset{1\times 12}{\lfloor N_w \rfloor} \\ k_{12} \underset{1\times 12}{\lfloor N_w \rfloor} \end{bmatrix}$$

(shell curvatures)

$$\underset{3\times 12}{[B_B]} = \begin{bmatrix} -\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\underset{1\times 12}{\lfloor N_w \rfloor}\right) \\ -\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\underset{1\times 12}{\lfloor N_w \rfloor}\right) \\ -2\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\underset{1\times 12}{\lfloor N_w \rfloor}\right) \end{bmatrix}$$

(bending)

ELASTIC STRAIN ENERGY

Elastic energy from membrane deformations

Elastic energy from bending deformations

$$U_e = U_e\left(\underset{3\times1}{\{\varepsilon\}}^{MID}\right) + U_e\left(\underset{3\times1}{\{\varkappa\}}\right)$$

$$U_e\left(\underset{3\times1}{\{\varepsilon\}}^{MID}\right) = \int_{A_e} \frac{1}{2} \underset{1\times3}{\lfloor \varepsilon \rfloor}^{MID} \cdot \underset{3\times1}{\{n\}}\, dA_e =$$

$$= \frac{1}{2}\int_{A_e} \underset{1\times20}{\lfloor q \rfloor} J_e \cdot \begin{bmatrix} \underset{8\times3}{[B_M]^T} \\ \underset{12\times3}{[B_S]^T} \end{bmatrix} \cdot \underset{3\times3}{[D_n]} \cdot \underset{3\times1}{\{\varepsilon^{MID}\}}\, dA_e =$$

ELASTIC STRAIN ENERGY (membrane)

$$U_e\left(\underset{3\times1}{\{\varepsilon\}^{MID}}\right) = \frac{1}{2}\underset{1\times20}{\lfloor q \rfloor_e} \cdot \int_{A_e} \begin{bmatrix} \underset{8\times3}{[B_M]^T} \\ \underset{12\times3}{[B_S]^T} \end{bmatrix} \underset{3\times3}{[D_n]} \cdot \left[\underset{3\times8}{[B_M]} \;\vdots\; \underset{3\times12}{[B_S]} \right] dA_e \cdot \underset{20\times1}{\{q\}_e} =$$

$$= \frac{1}{2}\lfloor q \rfloor_e \int_{A_e} \begin{bmatrix} \underset{8\times3}{[B_M]^T} \\ \underset{12\times3}{[B_S]^T} \end{bmatrix} \cdot \left[\underset{3\times3}{[D_n]} \cdot \underset{3\times8}{[B_M]} \;\vdots\; \underset{3\times3}{[D_n]} \cdot \underset{3\times12}{[B_S]} \right] dA_e \; \underset{20\times1}{\{q\}_e} =$$

$$= \frac{1}{2}\lfloor q \rfloor_e \int_{A_e} \left[\begin{array}{c|c} \underset{8\times3}{[B_M]^T} \underset{3\times3}{[D_n]} \underset{3\times8}{[B_M]} & \underset{8\times3}{[B_M]^T} \cdot \underset{3\times3}{[D_n]} \cdot \underset{3\times12}{[B_S]} \\ \hline \underset{12\times3}{[B_S]^T} \underset{3\times3}{[D_n]} \underset{3\times8}{[B_M]} & \underset{12\times3}{[B_S]^T} \underset{3\times3}{[D_n]} \cdot \underset{3\times12}{[B_S]} \end{array} \right] dA_e \; \underset{20\times1}{\{q\}_e}$$

ELASTIC STRAIN ENERGY (bending)

$$U_e\left(\underset{3\times1}{\{\varkappa\}}\right) = \int_{A_e} \frac{1}{2} \cdot \underset{1\times3}{\lfloor \varkappa \rfloor} \cdot \underset{3\times1}{\{m\}} \, dA =$$

$$= \frac{1}{2} \int_{A_e} \underset{1\times20}{\lfloor q \rfloor_e} \cdot \begin{bmatrix} \underset{8\times3}{[0]} \\ \underset{12\times3}{[B_B]^T} \end{bmatrix} \cdot \underset{3\times3}{[D_M]} \cdot \underset{3\times1}{\{\varkappa\}} \, dA =$$

$$= \frac{1}{2} \underset{1\times20}{\lfloor q \rfloor_e} \int_{A_e} \begin{bmatrix} \underset{8\times3}{[0]} \\ \underset{12\times3}{[B_B]^T} \end{bmatrix} \underset{3\times3}{[D_m]} \cdot \left[\underset{3\times8}{[0]} \;\vdots\; \underset{3\times12}{[B_B]} \right] dA_e \cdot \underset{20\times1}{\{q\}_e} =$$

ELASTIC STRAIN ENERGY (bending)

$$U_e\left(\underset{3\times1}{\{\chi\}}\right) = \frac{1}{2}\underset{1\times20}{\lfloor q \rfloor_e}\int_{A_e}\left[\begin{array}{c}\underset{8\times3}{[0]}\\ \underset{12\times3}{[B_B]^T}\end{array}\right]\cdot\left[\underset{3\times3}{[D_m]}\underset{3\times8}{[0]}\ \vdots\ \underset{3\times3}{[D_m]}\cdot\underset{3\times12}{[B_B]}\right]dA_e\cdot\underset{20\times1}{\{q\}_e} =$$

$$= \frac{1}{2}\lfloor q \rfloor_e\int_{A_e}\left[\begin{array}{c|c}\underset{8\times8}{[0]} & \underset{8\times12}{[0]}\\ \hline \underset{12\times8}{[0]} & \underset{12\times3}{[B_B]^T}\underset{3\times3}{[D_m]}\cdot\underset{3\times12}{[B_B]}\end{array}\right]dA_e\,\{q\}_e \Rightarrow$$

ELASTIC STRAIN ENERGY:

$$\Rightarrow U_e = \frac{1}{2} \underbrace{\lfloor q \rfloor_e}_{1\times 20} \cdot \underbrace{[k]_e}_{20\times 20} \cdot \underbrace{\{q\}_e}_{20\times 1}$$

where:

$$[k]_e = \int_{A_e} \left[\begin{array}{c|c} \underbrace{[B_M]^T}_{8\times 3} \underbrace{[D_b]}_{3\times 3} \underbrace{[B_M]}_{3\times 8} & \underbrace{[B_M]^T}_{8\times 3} \underbrace{[D_n]}_{3\times 3} \underbrace{[B_S]}_{3\times 12} \\ \hline \underbrace{[B_S]^T}_{12\times 3} \underbrace{[D_n]}_{3\times 3} \underbrace{[B_M]}_{3\times 8} & \underbrace{[B_S]^T}_{12\times 3} \underbrace{[D_n]}_{3\times 3} \underbrace{[B_S]}_{3\times 12} + \underbrace{[B_B]^T}_{12\times 3} \underbrace{[D_m]}_{3\times 3} \underbrace{[B_B]}_{3\times 12} \end{array} \right] dA_e$$

Shell element stiffness matrix

POTENTIAL ENERGY OF LOADING: $W_e = \underbrace{\lfloor q \rfloor_e}_{1\times 20} \cdot \underbrace{\{F\}_e}_{20\times 1}$

4-node shell element in Ansys

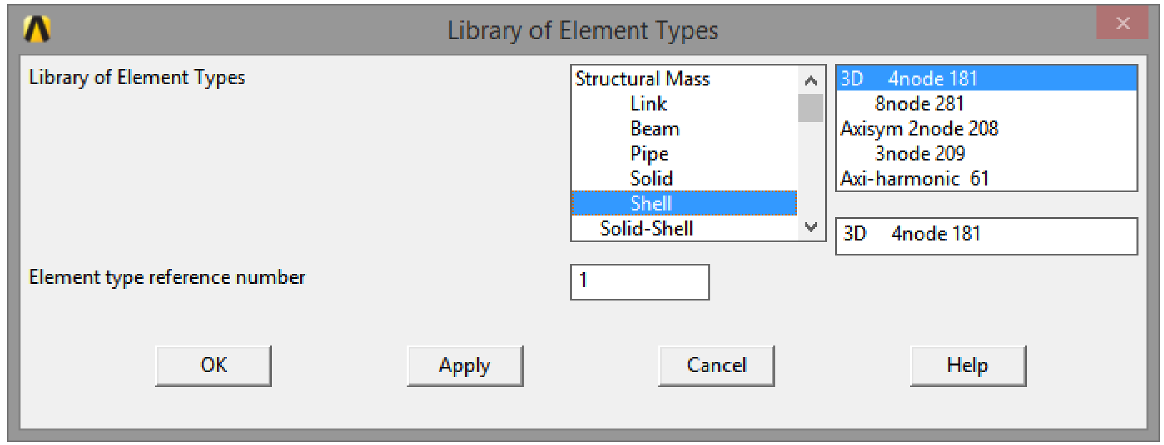
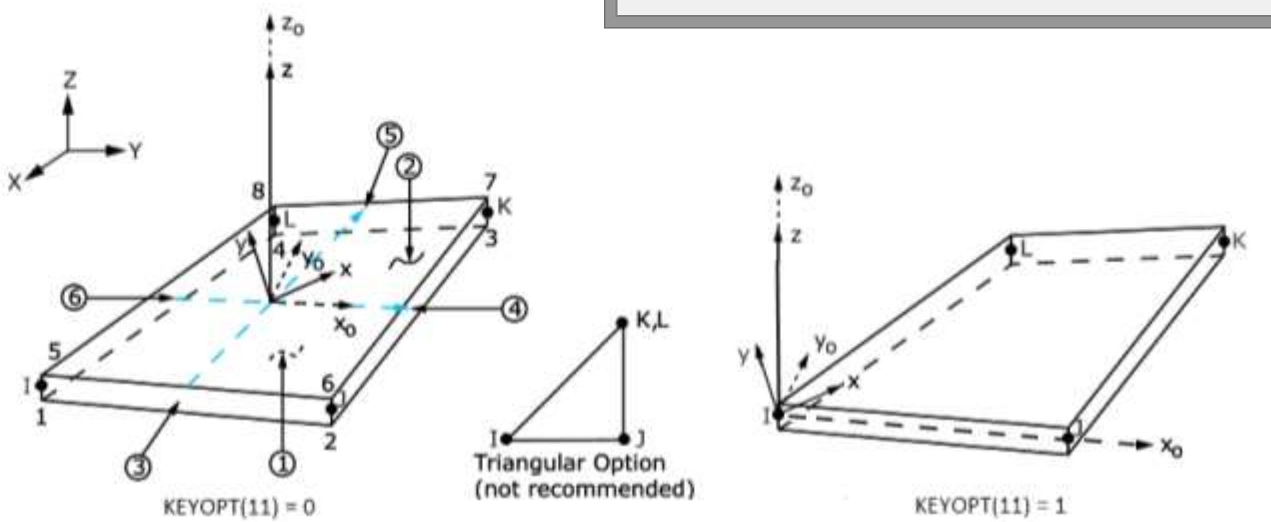


Figure 181.1: SHELL181 Geometry



x_0 = Element x axis if element orientation (**ESYS**) is not specified.

x = Element x axis if element orientation is specified.

8 node shell element in Ansys

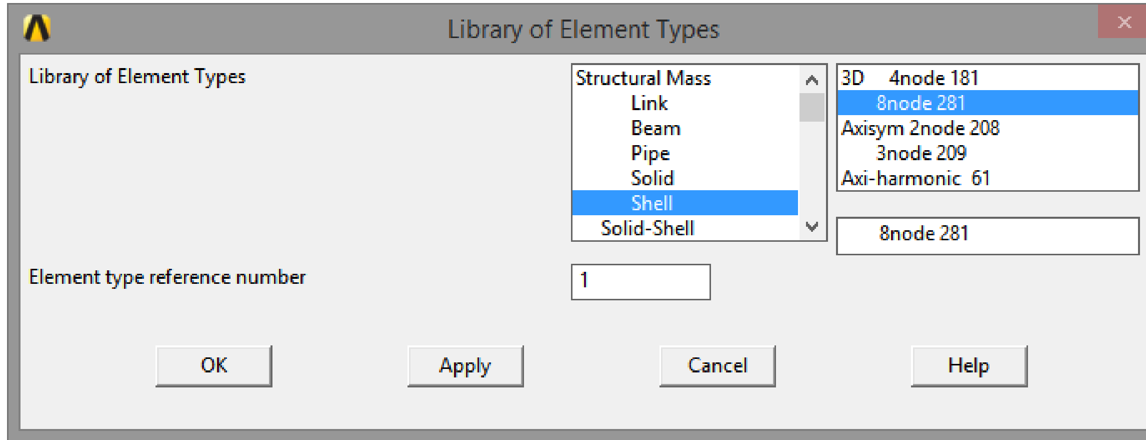
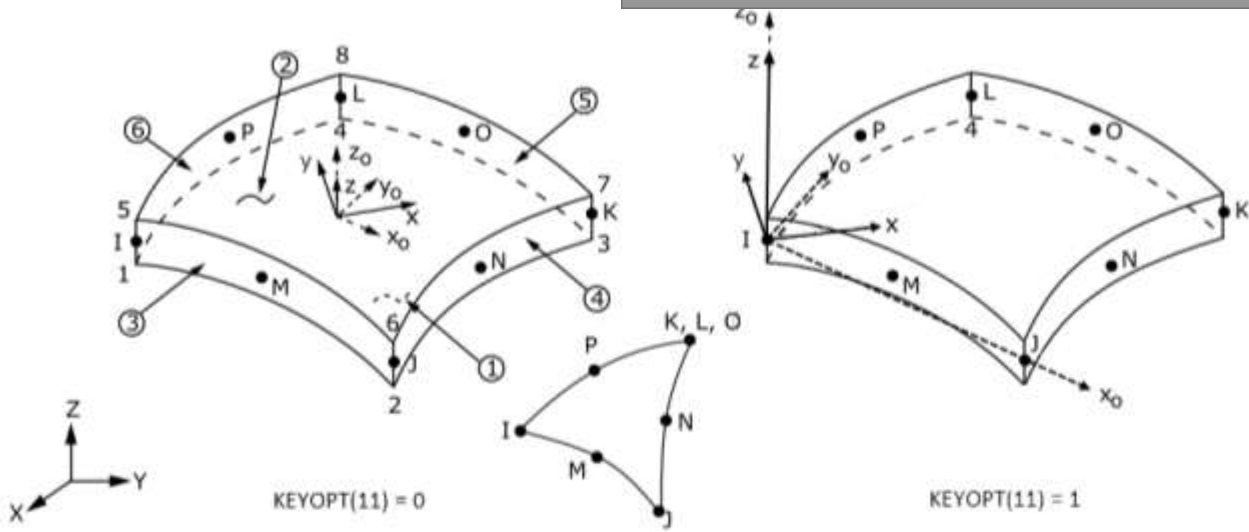


Figure 281.1: SHELL281 Geometry

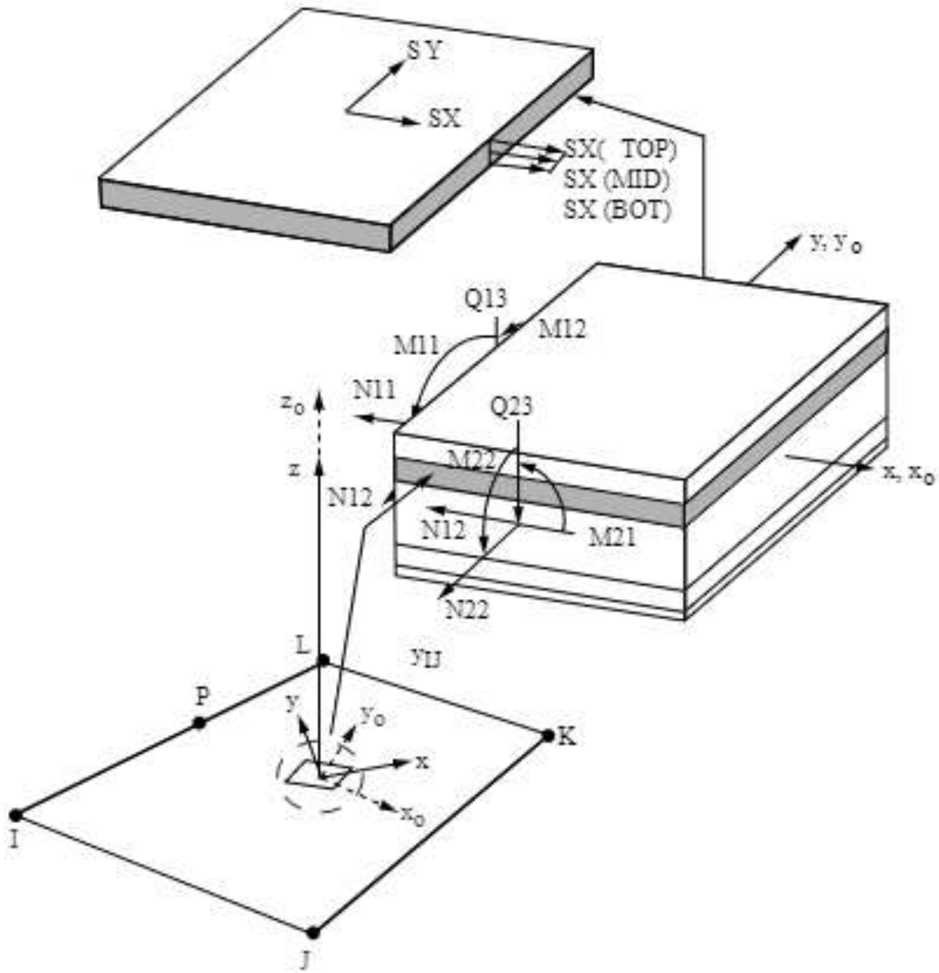


x_o = Element x axis if element orientation (**ESYS**) is not specified.

x = Element x axis if element orientation is specified.

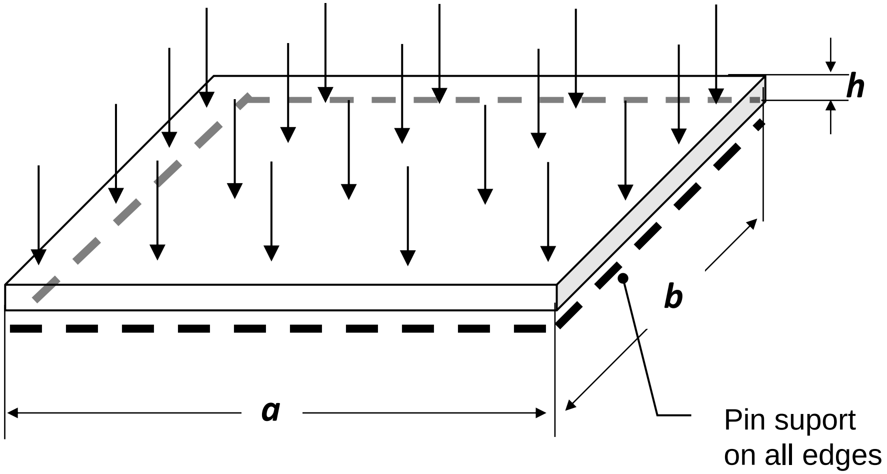
Layers option in shell element

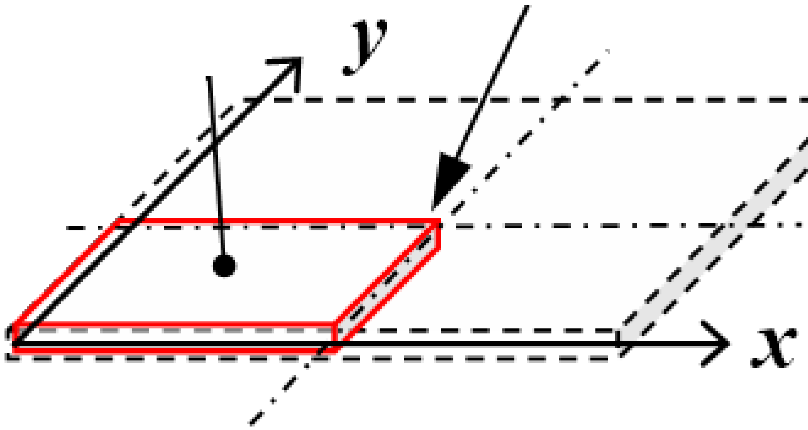
Figure 181.3: SHELL181 Stress Output

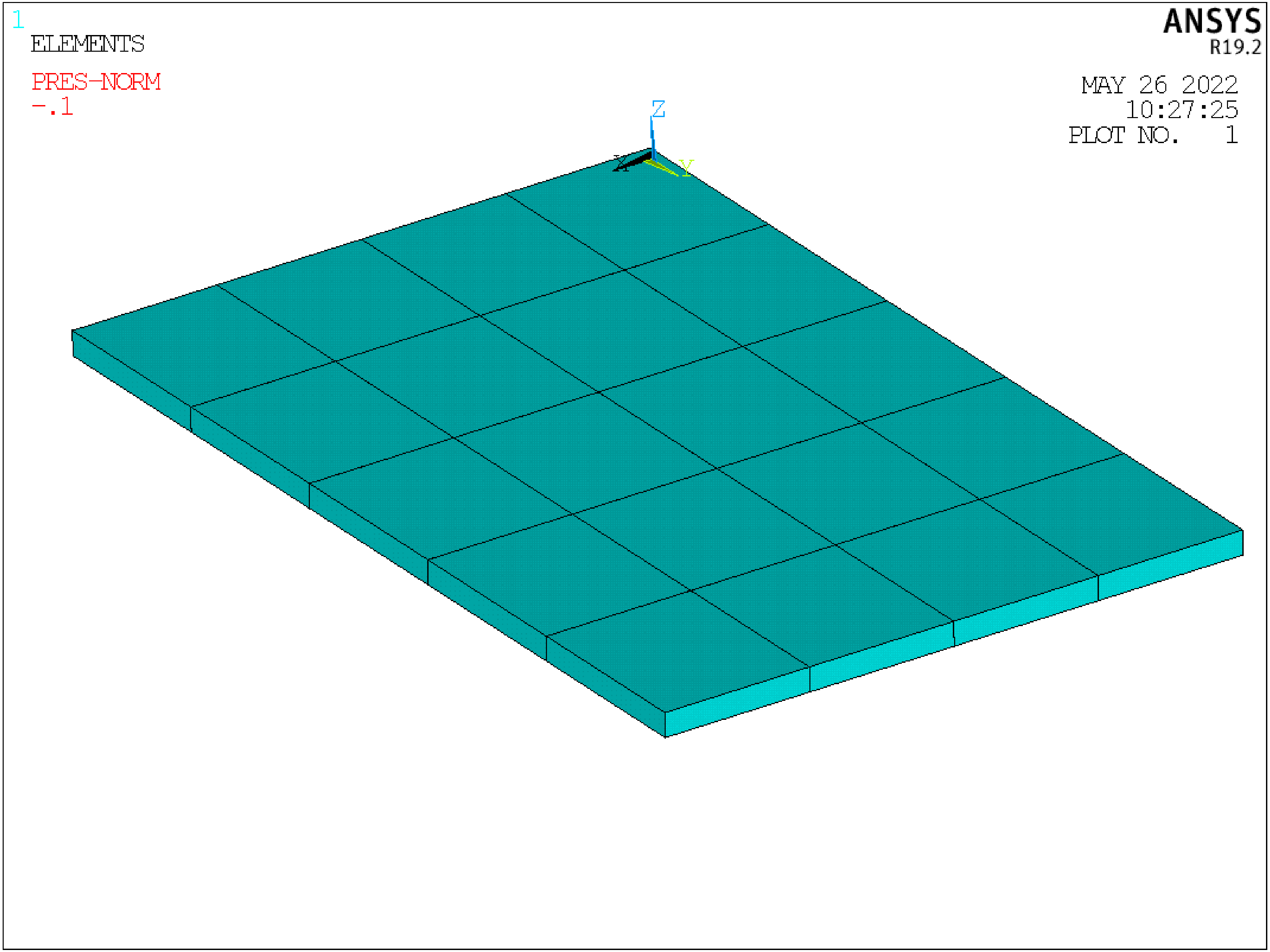


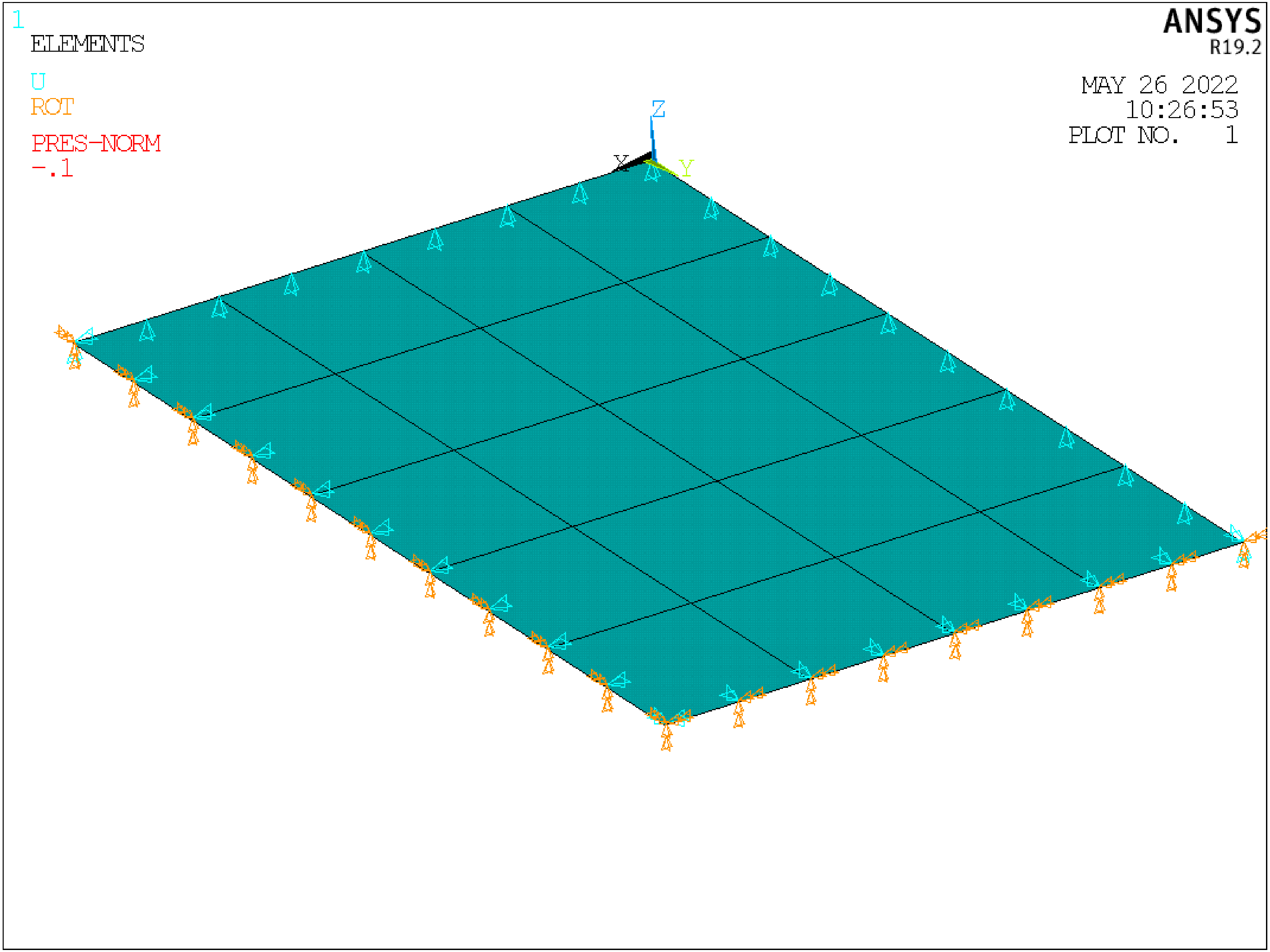
Bending of a rectangular plate

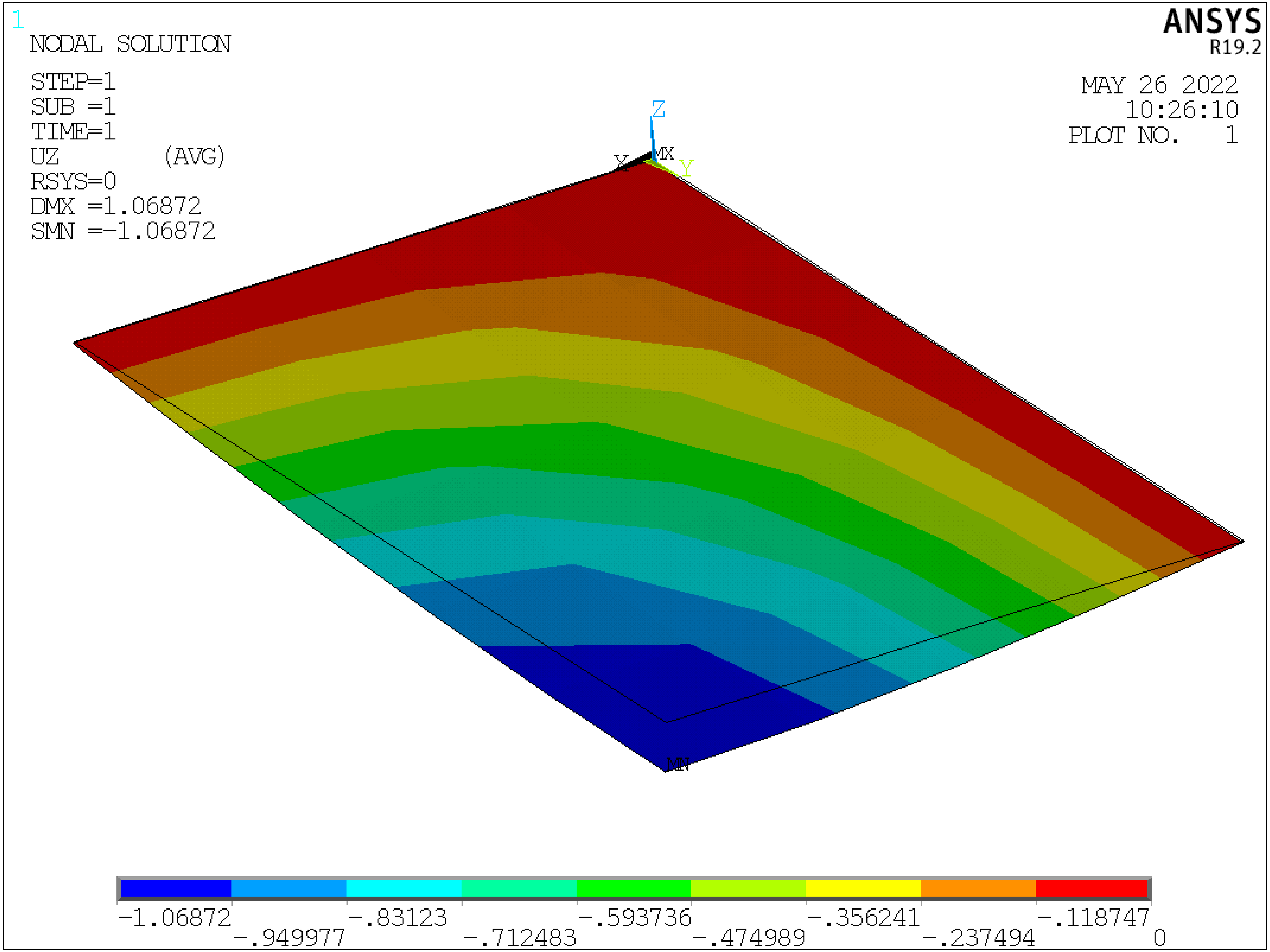
Data: $\boldsymbol{q}=0.1MPa$, $\boldsymbol{a}=200\ mm$, $\boldsymbol{b}=300mm$, $\boldsymbol{h}=4mm$, $\boldsymbol{E}=2\cdot10^5\ MPa$, $\boldsymbol{\nu}=0.3$

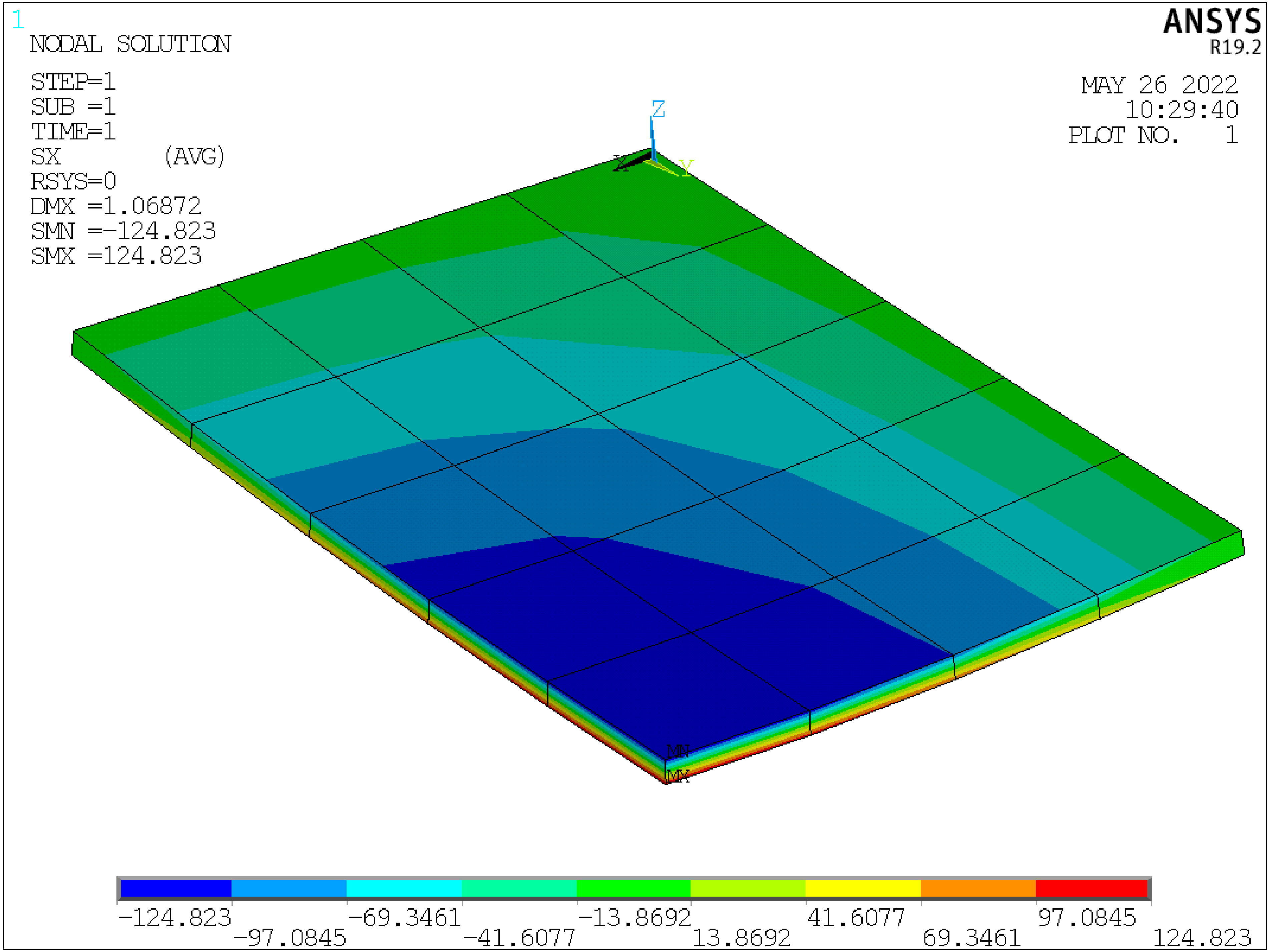




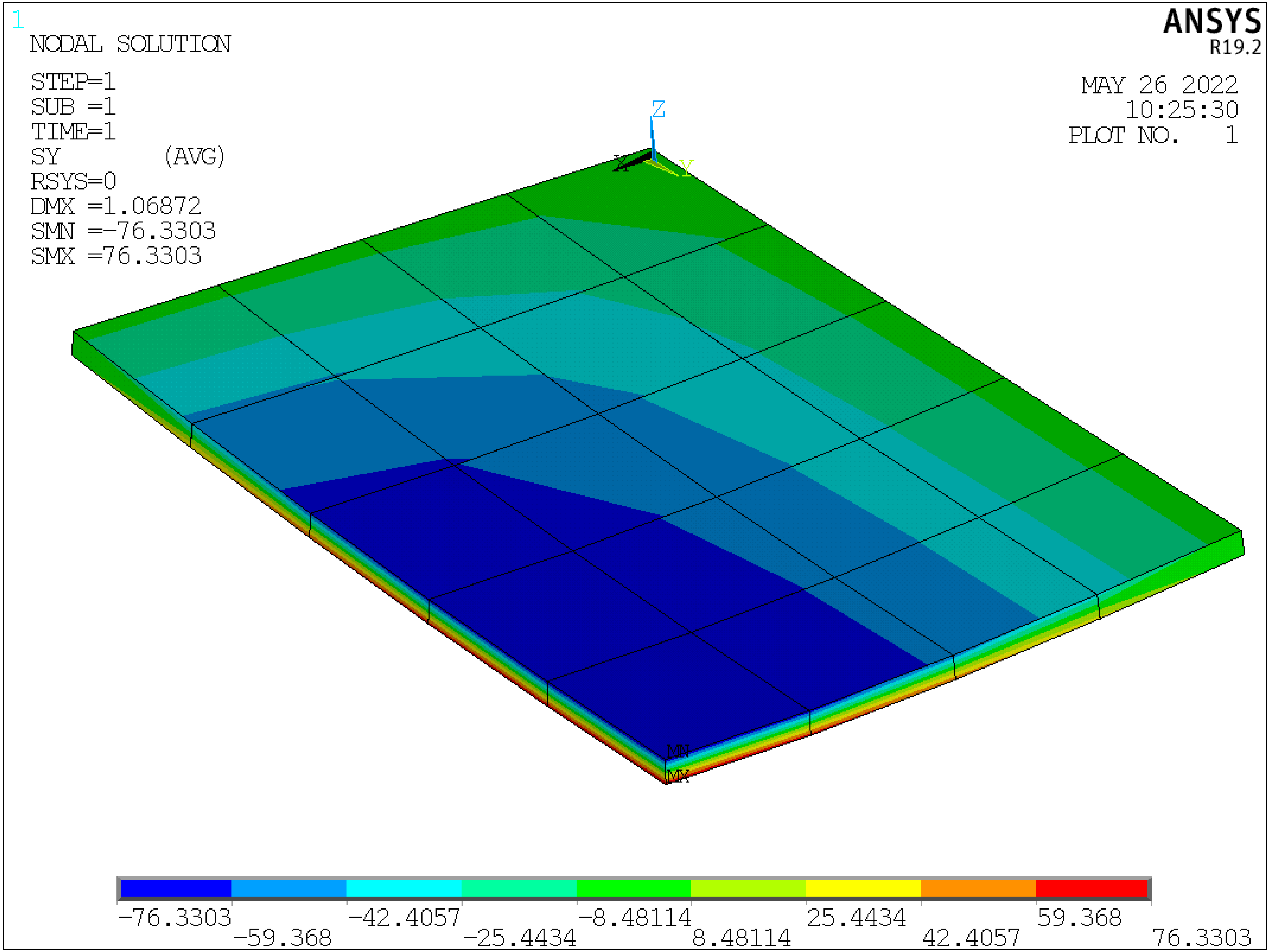

1
ELEMENTS
PRES-NORM
-.1
ANSYS
R19.2
MAY 26 2022
10:27:25
PLOT NO. 1
Z
Y
X


1
ELEMENTS
U
ROT
PRES-NORM
-.1
ANSYS
R19.2
MAY 26 2022
10:26:53
PLOT NO. 1
Z
X
Y

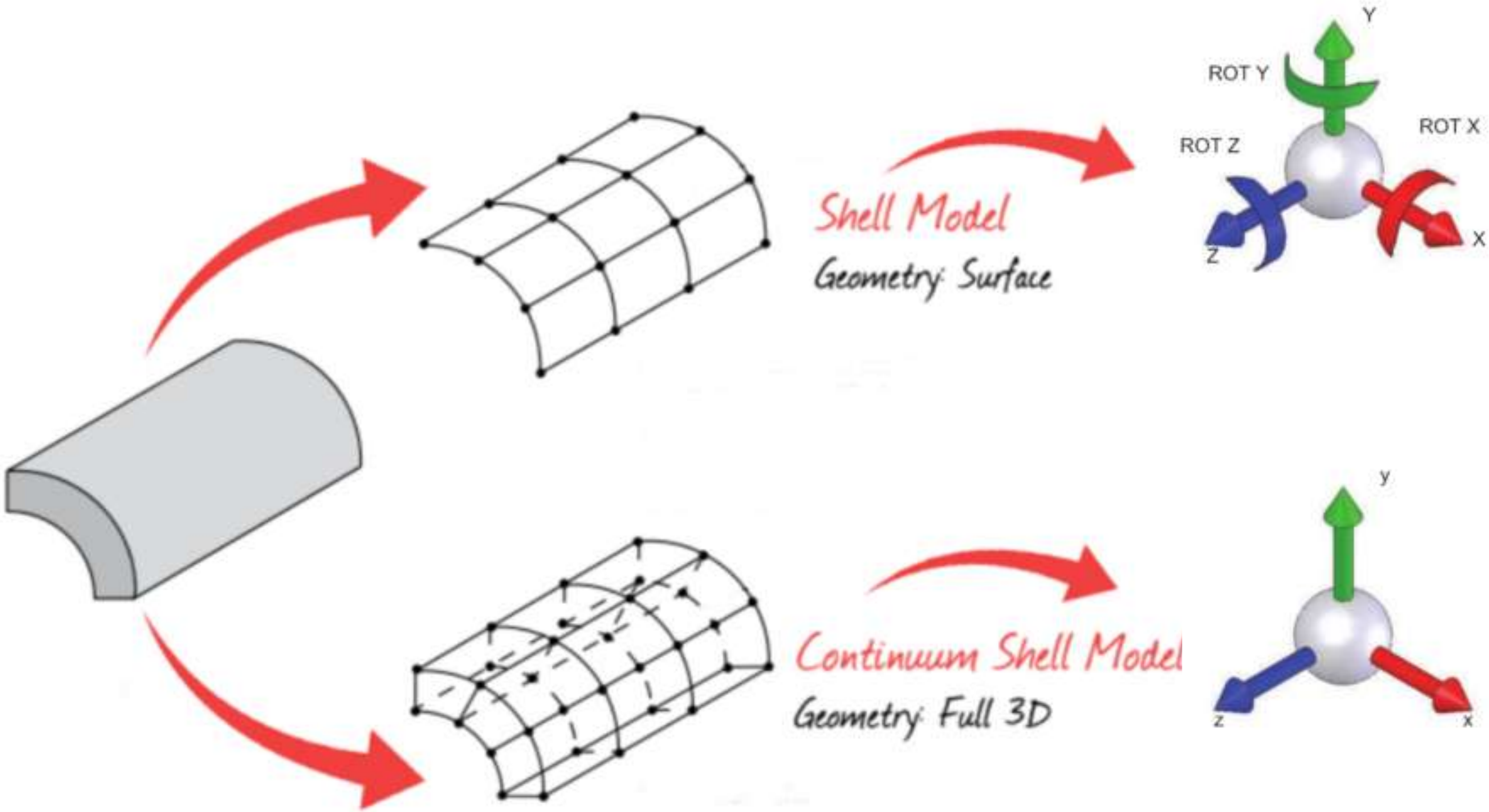

1
NODAL SOLUTION
STEP=1
SUB =1
TIME=1
UZ (AVG)
RSYS=0
DMX =1.06872
SMN =-1.06872
ANSYS
R19.2
MAY 26 2022
10:26:10
PLOT NO. 1
Z
Y
X
MX
MN
-1.06872
-.949977
-.83123
-.712483
-.593736
-.474989
-.356241
-.237494
-.118747
0



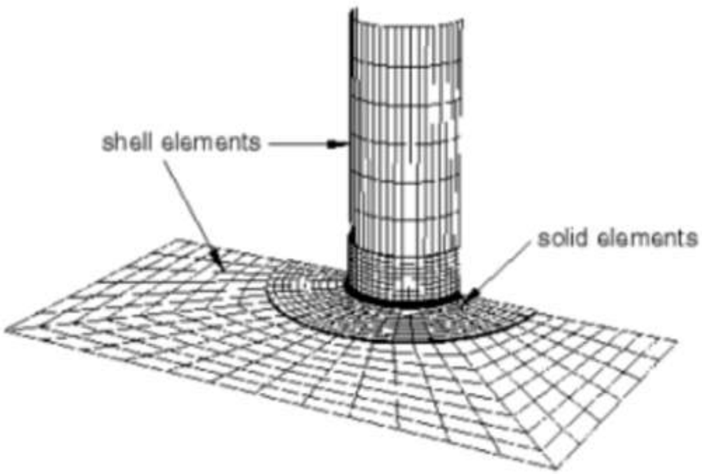
1
NODAL SOLUTION
STEP=1
SUB =1
TIME=1
SX (AVG)
RSYS=0
DMX =1.06872
SMN =-124.823
SMX =124.823
ANSYS
R19.2
MAY 26 2022
10:29:40
PLOT NO. 1
Z
X
Y
MN
MX
-124.823
-97.0845
-69.3461
-41.6077
-13.8692
13.8692
41.6077
69.3461
97.0845
124.823

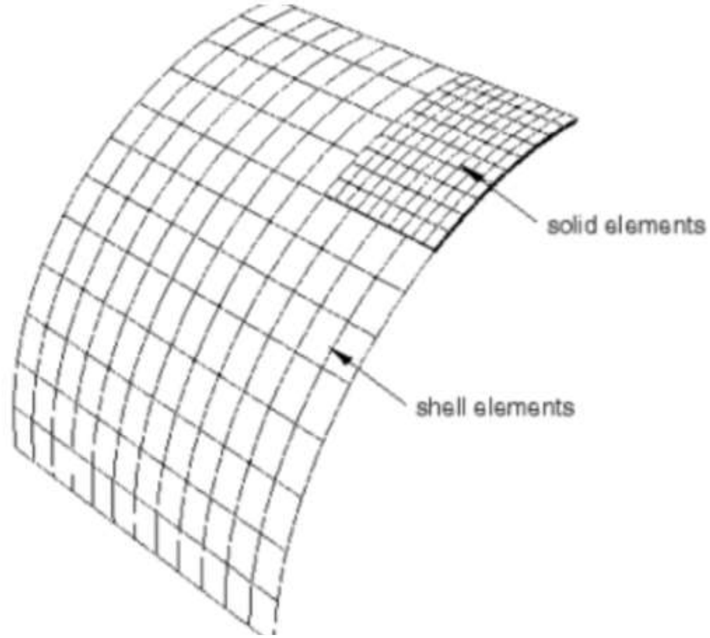

1
NODAL SOLUTION
STEP=1
SUB =1
TIME=1
SY (AVG)
RSYS=0
DMX =1.06872
SMN =-76.3303
SMX =76.3303
ANSYS
R19.2
MAY 26 2022
10:25:30
PLOT NO. 1
Z
X
Y
MN
MX
-76.3303
-59.368
-42.4057
-25.4434
-8.48114
8.48114
25.4434
42.4057
59.368
76.3303

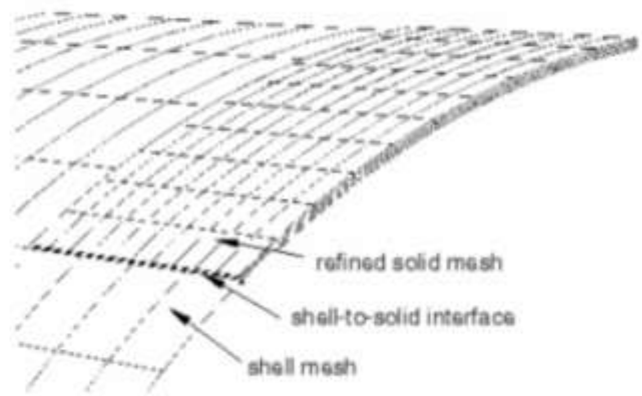
Comparison of the shell element with the Solid3D element



Connecting shell elements to Solid3D element







Modeling the wing structure